数理解析研究所講究録
531巻 1984年 18-32

Lacunas and Asymptotic Behaviors of Solutions for Hyperbolic Equations

双曲型方程式の解のLacunasと
漸近挙動について

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1. Introduction.

The concept of lacuna for the fundamental solution of a constant coefficient hyperbolic operator was defined by Atiyah-Bott-Gårding [1], in an attempt to clarify the results by Petrovsky [6]. They obtained a so-called Petrovsky condition, which seems very close, if not exactly equal, to a necessary and sufficient condition for a given domain in the propagation cone to become a lacuna.

We shall here pick up a rather simple case and consider a domain called the innermost conoid. The name is derived from the fact that, if the operator is a product of wave operators with different light speeds, the innermost conoid coincides with the cone surrounded by the wave surface of the slowest speed. In the case of the constant coefficient operators the innermost cone turns out to satisfy the Petrovsky condition.

Our advantage, however, is not only that the method provided here gives a simple proof for the innermost cone to become a lacuna, but that it is applicable to the case of strictly hyperbolic operators with variable coefficients.

A certain criterion of the strong lacuna (or the Huygens' principle) in the innermost conoid is an immediate consequence of our method. Although it is difficult in general to apply, it gives a simpler and more comprehensible account of classical Stellmacher's example ([7]).

We shall also discuss an asymptotic behavior of a solution with compactly supported initial data in the directions of rays lying in the innermost cone. The study of this kind was done by Littman and Lui [4] for some of the homogeneous, strictly hyperbolic, constant coefficient operators. Our results will encompass a larger class of operators including any homogeneous, constant coefficient, hyperbolic operator with the innermost cone and certain Euler-Poisson-Darboux type operators.

Most of the results stated in this note stems from the author's joint work with Kyril Tintarev ([5]). The rest will be included in forthcoming papers.

Let $P(t,x,D_{t,x})$ be a strictly hyperbolic operator of order m with the real analytic coefficients of $(t,x) = (t,x_1,\cdots,x_n)\in\mathbb{R}^{n+1}$ $(\theta=(1,0,\cdots,0)$ is the direction of the hyperbolicity). The k th fundamental solution $(k=0,\cdots,m-1)$ is defined by

$$(1.1)\qquad \begin{aligned} &PE_k = 0, \\ &\partial_t^j E_k = \delta_{jk}\delta(x-y), \quad t=0, \quad j = 0,1,\cdots,m-1. \end{aligned}$$

2. Innermost Cone.

We shall freeze the operator at $(0,y)$; $a(D)=P_m(0,y,D_{t,x})$.

Definition 2.4. The innermost cone of the operator $a(D)$ is defined by

$$(2.1) \qquad IC(a)=\{(t,x);t>0,\ X(t,x)\cap A(a)=\{0\}\},$$

where

$$X(t,x)=\{(\tau,\xi)\in\mathbb{R}^{n+1};\ t\tau+\langle x,\xi\rangle=0\},$$

and $A(a)$ is the set of real zeros of the polynomial $a(\tau,\xi)$.

Let us introduce some standard notation:

$$\Gamma(a,\theta)=\text{the component containing } \theta \text{ in } \mathbb{R}^{n+1}\setminus A,$$

$$K(a,\theta)=\{(t,x);\langle(t,x),\Gamma\rangle \geq 0\}.$$

The zeros of $a(\tau,\xi)=0$ are denoted by $\tau_j(\xi)$, $j=1,\cdots,m$. There is a permutation

$$(2.2) \qquad \begin{array}{l} j\to j' \quad \text{of } \{1,2,\cdots,m\} \text{ such that} \\ \tau_j(-\xi) = -\tau_{j'}(\xi). \end{array}$$

Lemma 2.2. Suppose $n\geq 2$. Then $IC(A)$ is an open, convex cone contained in $K(a,\theta)$.

Proof. For $(t,x)\in IC(a)$ the condition (2.1) can be written as

$$t>0,\ \tau_j(\xi)t+\langle x,\xi\rangle\neq 0,\quad \xi\in\mathbb{R}^n\setminus 0.$$

The left-hand side must therefore be always either >0 or <0.

Replacing ξ by $-\xi$ gives a similar inequality involving $\tau_{j'}(\xi)$ but with the opposite sign. A certain reordering of the zeros then yields

$$
\begin{aligned}
&\tau_j(\xi)t+\langle x,\xi\rangle > 0, \quad j=1,\cdots,m/2,\\
(2.3) \qquad &\tau_j(\xi)t+\langle x,\xi\rangle < 0, \quad j=m/2+1,\cdots,m\\
&t > 0, \quad j+j'=m+1, \text{ odd.}
\end{aligned}
$$

This separation of the zeros into two sets is made independently of the choice of $(t,x)\in IC(a)$. $IC(a)$ is therefore convex since it is the intersection of the convex sets of (t,x), each determined by one of the inequalities of (2.3). These inequalities are homogeneous in ξ, so $IC(A)$ is open as well.

To prove $IC(a)\subset K(a,\theta)$ we take $(t,x)\in IC(a)$ and suppose that $X(t,x)\cap\Gamma(a,\theta)\ni(\tau_0,\xi_0)$. Then for any real (τ,ξ) $a(s(\tau_0,\xi_0)+(\tau,\xi))=0$ must have real zeros of s. Choosing (τ,ξ) in $X(t,x)$ imphies $X\cap A \neq \{0\}$. This contradicts the definition (2.1). We therefore have $X(t,x)\cap\Gamma(a,\theta) = \emptyset$, which means that $\Gamma(a,\theta)$ lies in the same side of $X(t,x)$ as (t,x). In another word, $(t,x)\in K(a,\theta)$. Q.E.D.

As further properties we can prove

$IC(a)\cap\text{sing supp.}(E(a,\theta)) = \emptyset$,
$\partial IC(a)\subset\text{sing supp }(E(a,\theta))$, and that every point of $IC(a)$ satisfies the Petrovsky condition with respect to the operator $a(D)$.

Here $E(a,\theta)$ is the forward fundamental solution of $a(D)$;

$a(D)E=\delta(x)$, supp $E \subset \{(t,x); t \geqq 0\}$. We refer the definition of the Petrowsky condition to [1].

3. Innermost Conoid.

As for the operator $P(x,D)$ with variable coefficients its fundamental solutions $E_k(t,x,y)$ turn out to be real analytic along $IC(a)$ in an infinitesimal sense; more precisely, take any ray $(0,y)+s(t,x)$, $(t,x) \in IC(a)$, then there is a small $\varepsilon > 0$ such that E_k is real analytic on the segment $(0,y)+s(t,x)$, $0<s<\varepsilon$. It would be therefore natural to define the innermost conoid as follows. Let U be a small neighborhood of $(0,y)$:

(3.1) $IC(P,U)$=the connected component of $U \setminus$sing supp E_k containing the above segment.

We will now give two examples in which the inner most conoid is not empty.

<u>Example 3.1</u>. Let $P_m(0,y,0,\xi)$ be elliptic in ξ. (This is the case, for instance, when P is a product of wave operators.) The ellipticity implies that $(t,0) \in IC(a)$, $t > 0$, and so $IC(P,U)$ is not empty as well

<u>Example 3.2</u>. Let P be of second order. Write

$$a(\tau,\xi) = P_2(0,y,\tau,\xi) = \tau^2 - 2\langle\alpha,\xi\rangle + b(\xi).$$

The different zeros $\lambda_1(\xi)$, $\lambda_2(\xi)$ satisfy

$$\lambda_2 > \langle\alpha,\xi\rangle > \lambda_1.$$

This implies $(1,-\alpha) \in IC(a)$ and therefore $(0,y)+\varepsilon(1,-\alpha) \in IC(P,U)$ if $\varepsilon > 0$ is small enough.

4. An expression of E_k.

Our whole argument depends on a precise construction of fundamental solutions, including the computation of smooth residual terms that remain inexplicit in the usual analysis of singularities. Hamada's method [2], later applied to the real analytic case by Kawai [3], turns out to be useful for our purpose.

The principal part P_m has m differenct factors $\tau - a_\ell(t,x,\xi)$, $\ell=1,\cdots,m$ and each factor defines the phase function $\varphi_\ell(t,x,y,\xi)$ by

$$(4.1)\qquad \begin{aligned} &\partial_t \varphi_\ell - a_\ell(t,x,\partial_x \varphi_\ell) = 0, \\ &\varphi_\ell(0,x,y,\xi) = \langle x-y,\xi \rangle. \end{aligned}$$

We shall also use

$$(4.2)\qquad \begin{aligned} \chi_s(z) &= \Gamma(-s)e^{-i\pi s}z^s \ , \quad s=-1,-2,\cdots \\ &= z^s(-\log z + c_s + \pi i)/s!, \quad s=0,1,\cdots, \end{aligned}$$

where $-\pi < \arg z < \pi$, $c_0 = \Gamma'(1)$, $c_s = s^{-1} + c_{s-1}$. The relationship $d\chi_s/dz = \chi_{s-1}$ is the key property that makes them useful in constructing the fundamental solutions.

E_k can be written

$$(4.3)\qquad E_k = (2\pi i)^{-n} \sum_{\ell=1}^{m} \sum_{s=0}^{\infty} \int_{|\xi|=1} p_\ell^{(s)}(x,y,\xi)\chi_{s-n}(\varphi_\ell + i0)\omega(\xi),$$

where

$$\omega(\xi)=\sum(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\cdots\wedge d\xi_n .$$

The functions $p_\ell^{(s)}$, $\ell=1,\cdots,m$, $s=0,1,\cdots$, are obtained by solving the so-called transport equations. They prove to be holomorphic in a common neighborhood of $(0,y)\times S^n$. The series (4.3) converges to the distribution E_k.

The permutation (2.2) now induces

$$\begin{aligned} &\varphi_\ell(t,x,y,-\xi)=-\varphi_{\ell'}(t,x,y,\xi) \\ (4.4)\qquad &p_\ell^{(s)}(t,x,y,-\xi)=(-1)^s p_{\ell'}^{(s)}(t,x,y,\xi). \end{aligned}$$

Besides,

$$\chi_s(-\varphi+i0)=(-1)^s\chi_s(\varphi-i0), \quad s=-1,-2,\cdots$$

$$\chi_s(-\varphi+i0)=(-1)^s\chi_s(\varphi-i0)-(i\pi)(-\varphi)^s/s! \qquad s=1,2,\cdots.$$

Therefore we get

$$\begin{aligned} &\int p_{\ell'}^{(s)}\chi_{s-n}(\varphi_{\ell'}+i0)\omega(\xi) \\ &=\int p_\ell^{(s)}(-1)^n\chi_{s-n}(\varphi_\ell-i0)\omega(\xi)-(-1)^n i\pi\int p_\ell^{(s)}\varphi_\ell^{s-n}/(s-n)!\,\omega(\xi) \end{aligned}$$

if $s-n=0,1,\cdots$ and a similar fomula without the additional term if $s-n\neq 0,1,\cdots$.

Now suppose that $IC(a)$ is not empty. Then by (2.3) m is even and we get the following useful expression.

$$E_k=\sum_{\ell=1}^{m/2}E_{k\ell}+R_k,$$

$$(4.5)\qquad E_{k\ell}=(2\pi i)^{-n}\sum_{s=k}^{\infty}\int p_\ell^{(s)}g_{s-n}(\varphi_\ell)\omega(\xi)/(s-n)!,$$

$$R_k = 2^{-n}(-i\pi)^{-n-1}\sum_{\ell=1}^{m/2}\sum_{s=n}^{\infty} p_\ell^{(s)}\varphi_\ell^{s-n}\omega(\xi)/(s-n)!,$$

where $g_s(x) = \chi_s(x+i0)+(-1)^n\chi_s(x-i0)$. R_k turns out to be real analytic in a neighborhood of $(0,y)$.

5. Lacunas.

The expression (4.5) immediately brings about a result that the innermost conoid is a weak lacuna.

Theorem 5.1. Let P be a strictly hyperbolic operator with real analytic coefficients defined near a point $(0,y)\in \mathbb{R}^{n+1}$. Assume that n is odd and the innermost cone $IC(a)$ $(a=P_m(0,y,\tau,\xi))$ is not empty. Then there is a small neighbourhood U of $(y,0)$ such that the fundamental solutions E_k, $k=1,\cdots,m$, restricted to $IC(P,U)$ can be extended to real analytic functions in a neighborhood of $(y,0)$.

Proof. Let $\tau_\ell(\xi)$ be a zero of $a(\tau,\xi)$. If $(t,x-y)\in IC(a)$, then from (2.3)

$$\tau_\ell(\xi)t + \langle x-y,\xi\rangle > 0,\ \ell=1,\cdots,m/2,\ t>0,\ \xi\in\mathbb{R}^n.$$

Since $\varphi_\ell(t,x,y,\xi)=\langle x-y,\xi\rangle+\tau_j(\xi)t+o((|x-y|+t))$, $\varphi_\ell > 0$ on a small segment $(0,y)+s(t,x-y)$, $0<s<\varepsilon$. On the other hand $g_{s-n}(\varphi_\ell)=0$ for $\varphi_\ell > 0$ and n odd. Thus, by (4.5), $E_k=R_k$ with (t,x) in a neighborhood of the segment. Therefore the analytic extension of $E_k|_{IC(P,U)}$ to a neighborhood of the origin is given by R_k itself.

The theorem, combined with Example 3.2, gives

Corollary 5.2. Let P be an operator of second order satisfying the condition of Theorem 5.1. Then, in a neighborhood of $(0,y)$, the inside of the propagation cone of E_k $(k=0,1)$ becomes a weak lacuna.

On the other hand when the space dimension n is even we can not expect the innermost conoid to become a lacuna.

Proposition 5.3. Suppose that the condition of Theorem 5.1 holds except that n is here assumed to be even. Then the function $E_k|_{IC(P,U)}$ has no smooth extension to $(0,y)$.

Proposition 5.3 and the proof of Theorem 5.1 lead to a criterion for the strong lacuna.

Proposition 5.4. Let U be a sufficiently small neighborhood of $(0,y)$. $IC(P,U)$ becomes a strong lacuna for E_k, i.e. E_k vanishes in $IC(P,U)$, if and only if

i) n is odd and

ii) $R_k = 0$.

The second condition is equivalent to $\partial_t^j R_k=0$, $j=0,\cdots,m-1$ because R_k is itself a solution of $Pu=0$.

There are a few examples in which the above criterion turns out to be of some use. Before stating them we shall give a remark; if P_m has the constant coefficients and if $IC(P,U)$ is a lacuna for every y in an open neighborhood; then one gets the following condition equivalent to (ii)

(4.6) $$\sum_{j=0}^{h}\sum_{\ell=1}^{m/2}\int \binom{h}{j}\partial_t^j p_\ell^{(s+h-j)}(0,x,\xi)\lambda_\ell^{h-j}(\xi)\xi^\alpha\omega(\xi)=0,$$

$h=0,\cdots,m-1,\quad |\alpha|=s-n,\quad s=n,n+1,\cdots.$

<u>Example 5.5.</u> Let $P=P_m$ is a constant coefficient operator. Then

$$E_k = (2\pi i)^{-n}\sum_{\ell=1}^{m}\int p_\ell^{(k)}(\xi)g_{k-n}(<x-y,\xi>+\lambda_\ell(\xi)t)\omega(\xi).$$

In particular, $p^{(s)}=0$, $s > k$. Therefore, if n is odd, IC(P) is a strong lacuna for $k < n$. For $k \geqq n$, E_k restricted to the (weak) lacuna is a polynomial in (t,x,y) of order $k-n$.

<u>Example 5.6.</u> (Euler-Poisson-Darboux type) Consider the following Cauchy problem

(4.7) $$\begin{aligned}&(\partial_t^2 - \Delta + \frac{\lambda}{(1+t)^2})E_k = 0, \quad t \geqq 0\\ &\partial_t^j E_k = \delta_{jk}\delta(x-y), \quad t=0,\ j=0,1.\end{aligned}$$

There are two phase functions $\varphi^{\pm} = \pm t|\xi|+<x-y,\xi>$ and we denote the corresponding amplitude functions by $\{p_\pm^{(s)}\}$ for E_0 and by $\{\tilde{p}_\pm^{(s)}\}$ for E_1. Then a little involved computation shows

$$p_+^{(s)}=(-1)^s p_-^{(s)} = 2^{-s-1}\sum_{j=0}^{s}(-1)^{s-j}C_j(\lambda)\bar{C}_{s-j}(\lambda)/(1+t)^j$$

$$\tilde{p}_+^{(s)} = (-1)^s \tilde{p}_-^{(s)} = 2^{-s} \sum_{j=0}^{s-1} (-1)^{s-1-j} C_j(\lambda) C_{s-1-j}(\lambda)/(1+t)^j$$

where

$$C_0(\lambda) = \bar{C}_0(\lambda) = 1$$

$$C_j(\lambda) = (j!)^{-1}\lambda(\lambda+2)\cdots(\lambda+(j-1)j)$$

$$\bar{C}_j(\lambda) = (j!)^{-1}\lambda(\lambda+2)\cdots(\lambda+(j-2)(j-1))(\lambda-(j-1)j),$$
$$j=1,2,\cdots.$$

Now it follows that, if $\lambda = -\nu(\nu+1)$ for some positive integer ν, then $p_{\pm}^{(2\nu+1)} \neq 0$ and $\tilde{p}_{\pm}^{(2\nu+1)} \neq 0$, while $p_{\pm}^{(s)} = \tilde{p}_{\pm}^{(s)} = 0$ for all $s \geq 2\nu+2$.

Proposition 5.7. Let E_0 (resp. E_1) be a solution of (4.7) and U be an arbitrary open set in $\mathbb{R}^n$. Then

$$\text{supp } E_0 \text{ (resp.} E_1) \supset \{(t,x) \in \mathbb{R}^{n+1};\ |x-y| = t \geq 0\},$$

$$y \in U$$

if and only if n is odd and $\lambda = -\nu(\nu+1)$ for some non-negative integer ν such that $\nu \leq (n-3)/2$.

The 'if' part follows easily from the above remark. The 'only if' part can be proved by using the formula (4.6).

It is possible to look at the example 5.6 from a more general point of view, as was done by Stellmacher [7]. We will now present it in a slightly geometrical framework.

Proposition 5.8. Let $M = \{x \in \mathbb{R}^{n+1};\ x_j > 0,\ j=0,\cdots,n\}$ be given a structure of Lorenzian manifold by the pseudo-Riemannian metric $g_{ij} = (\prod_0^n x_k^{\lambda_k})\tilde{\delta}_{ij}$, $\lambda_k \in \mathbb{R}$, where $\tilde{\delta}_{ij} = -1$, 1 or 0 according to $i=j=0$, $i=j\geqq 1$ or $i\neq j$ respectively. In order that the Huygens' principle is valid on M, i.e. that the wave equation $\Box_M$ on M provides the strong lacuna inside the propagation cone of the fundamental solutions E_k, $k=0,1$, for any y, the following condition is necessary and sufficient: i) n is odd and ii) either $-\frac{n-1}{4}\lambda_i$ or $\frac{n-1}{4}\lambda_i-1$ becomes a non-negative integer ν_i such that $\sum_0^n \nu_i \leqq \frac{n-3}{2}$.

6. Asymptotic Behaviors.

Littman and Lui [4] discovered the following interesting asymptotic behavior of a solution for the wave equation: Let u be a solution of

$$(\partial_t^2-\Delta)u = 0$$

$$u=u_0,\ \partial_t u=u_1,\ t=0,\ u_0,u_1 \in C_0^\infty(\mathbb{R}^n),$$

then i) u has the asymptotic expansion

$$u(t,0) \sim \sum_{j=0}^\infty a_j t^{1-n-j}, \quad t \to \infty ,$$

ii) if n is even and $u(t,0)=O(t^{-\infty})$, $t \to \infty$, then $u(t,0)\equiv 0$, $t\geqq 0$.

This example made up a core in their work where they drew a similar conclusion for the strictly hyperbolic, constant-coefficient, homogeneous, operators $P(D_t,D_x)$ with elliptic $P(0,\xi)$.

We can generalize their result to the case where $P(D)$ is assumed only to be a constant-coefficient, homogeneous, hyperbolic operator having the innermost conoid. Then the asymptotic behavior at i) and ii) is replaced by similar ones taken in the direction of any ray lying in the cone.

In what follows we will state another generalization of the above example. The Euler-Poisson-Darboux equation is again considered;

$$(6.1)\qquad \begin{aligned} &\left(\partial_t^2 - \Delta + \frac{\lambda}{(1+t)^2}\right)u = 0 \\ &u = u_0,\ \partial_t u = u_1,\ t = 0,\ u_j \in C_0^\infty. \end{aligned}$$

We already know the exact form of the fundamental solutions, and when $\lambda = -\nu(\nu+1)$ with some non-negative integer ν, the expression (4.5) has only a finite number of terms and therefore is valid globaly. In this case it is not difficult to compute the asymptotic expansion of u. In the following propositions we always assume the above condition of λ.

Proposition 6.1. For any $x_0 \in \mathbb{R}^n$ and $0 < \delta < 1$ there are holomorphic functions $a_j(w)$, $b_j(w)$ defined in the open unit disc of $\mathbb{C}^n$, such that

$$u(x_0+\alpha t,t) \sim \sum_{j=0}^{\infty}(a_j(\alpha)+b_j(\alpha)\log t)t^{\nu+1-n-j},$$

$$|\alpha| < \delta,\quad \alpha\in\mathbb{R}^n,\quad (\nu \geqq (n-1)/2)$$

$$\sim \sum_{j=0}^{\infty} a_j(\alpha)t^{\nu+1-n-j},\quad |\alpha| < \delta\quad (\nu<(n-1)/2),$$

as $t \to \infty$.

Proposition 6.2. Let $\alpha \in \mathbb{R}^n$, $|\alpha| < 1$. The following asymptotic expression holds with some polynomials $a_j'(x)$ and $b_j'(x)$ of order up to j; for any conpact set K of $\mathbb{R}^n$ one has

$$u(x+\alpha t,t) \sim \sum_{j=0}^{\infty} (a_j'(x)+b_j'(x)\log t)t^{\nu+1-n-j}, \quad x \in K,$$

$$(\nu \geqq (n-1)/2)$$

$$\sim \sum_{j=0}^{\infty} a_j'(x)t^{\nu+1-n-j}, \quad x \in K \qquad (\nu < (n-1)/2)$$

as $t \to \infty$.

Proposition 6.3. Let n be even and $\nu < \frac{n-1}{2}$. If $u(x_0+\alpha t,t) = o(t^{-\infty})$, $t \to \infty$ with some $\alpha \in \mathbb{R}^n$, $|\alpha| < 1$, then $u(x_0+\alpha t,t) \equiv 0$, $t \geqq 0$.

This proposition generalizes (ii) of Littman-Lui's example. It was also shown in [4] that if u decays faster than any negative power of t in a semi-infinite cylinder as $t \to \infty$, then u vanishes identically, provided that the initial data of u have compact sypport. We will be able to give an affirmative answer in our case as well.

Proposition 6.4. Let n and ν satisfy the condition of Proposition 6.3. Let $\alpha \in \mathbb{R}^n$ with $|\alpha| < 1$. If a solution u of (6.1) has the asymptotic behavior $u(y+\alpha t,t) = o(t^{-\infty})$ as $t \to \infty$ for any y in an open set of $\mathbb{R}^n$ then it follows that $u_0 = u_1 = 0$.

Proof. The hypothesis implies that in Proposition 6.2

$a_j'(y) = 0$, $j=0,1,\cdots$, when y lies in the open set (we have $\nu < (n-1)/2$ in this case.) Since they are polynomials, it follows that $a_j'=0$. This in turn means that $u(t,x+\alpha t)=o(t^{-\infty})$ for all $x \in \mathbb{R}^n$. Proposition 6.3 enables us to conclude that $u(t,x+\alpha t)=0$, $t \geqq 0$, i.e. $u(t,x)=0$ for all (t,x), $t \geqq 0$. Therefore, obviously $u_0 = u_1 = 0$. This completes the proof.

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